Neutrino spin-flavor oscillations in electromagnetic fields of various configurations

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Abstract

We study spin-flavor oscillations of Dirac neutrinos with mixing and having non-zero matrix of magnetic moments in magnetic fields of various configurations. We discuss constant transversal and twisting magnetic fields. To describe the dynamics of Dirac neutrinos we use relativistic quantum mechanics approach based on the exact solutions to the Dirac-Pauli equation in an external electromagnetic field. We derive transition probabilities for different neutrino magnetic moments matrices.

1 Introduction

Neutrino conversions from one flavor to another combined with the change of the particle helicity, e.g. $\nu_e^{\mathrm{L}} \leftrightarrow \nu_\mu^{\mathrm{R}}$, are usually called neutrino spin-flavor oscillations (see Refs. [1]). This neutrino oscillations type is important since it could be one of the possible explanations of the time variability of the solar neutrino flux (see, e.g., Refs. [2]). However it was suggested in Refs. [3] that neutrino spin-flavor oscillations in solar magnetic fields give a sub-dominant contribution in to the total conversion of solar neutrinos.

In this paper we summarize the results of our recent studies (see Refs. [4, 5]) of neutrino spin-flavor oscillations in external electromagnetic fields of various configurations. We suppose that neutrinos are Dirac particles. To describe the evolution of the neutrino system we use the approach based on the relativistic quantum mechanics. We start from exact solutions to the Dirac-Pauli equation in an external magnetic field and then derive the neutrino wave functions satisfying the given initial condition. We used this method to describe neutrino flavor and spin-flavor oscillations in vacuum and in various external fields (see Refs. [4–7]). Note that neutrino neutrino spin-flavor oscillations in electromagnetic fields of various configurations were examined in Refs. [8] using the standard quantum mechanical approach. The propagation and oscillations of neutrinos in strong magnetic fields was also studied in Refs. [9].

In Sec. 2 we formulate the initial condition problem for the system of two Dirac neutrinos which mix and have non-vanishing matrix of magnetic moments. Moreover the mass and the magnetic moments matrices are assumed to independent since we study this system on the phenomenological level. It means that the diagonalization of the mass matrix does not involve the diagonalization of the magnetic moments matrix. In Sec. 3 on the basis of the known solution to the Dirac-Pauli equation in the constant transversal magnetic field we describe the time evolution of the system in question. We obtain the most general final neutrino wave function which exactly takes into account all neutrino magnetic moments and valid for arbitrary strength of the external magnetic field. Then we discuss several applications of the yielded results and

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calculate the transition probability for various magnetic moments matrices. In Sec. 4 we apply the same technique for the analysis of spin-flavor oscillations of Dirac neutrinos in the twisting magnetic field. We start from the recently obtained solution to the Dirac-Pauli equation for this magnetic field configuration. We derive transition probabilities for various types of neutrino magnetic moments matrices. Then in Sec. 5 we summarize our results.

2 Electrodynamics of mixed neutrinos with magnetic moments

Let us study the evolution of two Dirac neutrinos (ν_α, ν_β) that mix and interact with the external electromagnetic field $F_{\mu\nu} = (\mathbf{E}, \mathbf{B})$. The Lagrangian for this system has the form

$$\mathcal{L}(\nu_\alpha, \nu_\beta) = \sum_{\lambda=\alpha,\beta} \bar{\nu}_\lambda \mathrm{i}\gamma^\mu \partial_\mu \nu_\lambda - \sum_{\lambda\lambda'=\alpha,\beta} \left[m_{\lambda\lambda'} \bar{\nu}_\lambda \nu_{\lambda'} + \frac{1}{2} M_{\lambda\lambda'} \bar{\nu}_\lambda \sigma_{\mu\nu} \nu_{\lambda'} F^{\mu\nu} \right], \tag{1}$$

where $\sigma_{\mu\nu} = (\mathrm{i}/2)(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$. The neutrino mass matrix $(m_{\lambda\lambda'})$ and the neutrino magnetic moments matrix $(M_{\lambda\lambda'})$ are generally independent.

To describe the dynamics of the system we set the initial condition by specifying the initial wave functions of flavor neutrinos ν_λ and then analytically determine the wave functions at subsequent moments of time. We assume that the initial condition is

$$\nu_\alpha(\mathbf{r}, 0) = 0, \quad \nu_\beta(\mathbf{r}, 0) = \xi(\mathbf{r}), \tag{2}$$

where $\xi(\mathbf{r})$ is a given function. Let us choose it in the following form: $\xi(\mathbf{r}) = e^{\mathrm{i}\mathbf{kr}}\xi_0$, where $\mathbf{k} = (k, 0, 0)$ and $\xi_0^\mathrm{T} = (1/2)(1, -1, -1, 1)$. One can check that $(1/2)(1 - \Sigma_1)\xi_0 = \xi_0$, i.e. the wave function $\xi(\mathbf{r})$ corresponds to a left-handed relativistic neutrino propagating along the x-axis. The similar choice of the initial condition was adopted in our works [4–7].

In order to diagonalize the mass matrix in Eq. (1) we introduce the mass eigenstates wave functions, ψ_a, $a = 1, 2$, obtained from the original flavor wave functions ν_λ through the unitary transformation

$$\nu_\lambda = \sum_{a=1,2} U_{\lambda a} \psi_a, \tag{3}$$

where the matrix $(U_{\lambda a})$ is parametrized with help of the mixing angle θ,

$$(U_{\lambda a}) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4}$$

The Lagrangian (1) rewritten in terms of the fields ψ_a takes the form

$$\mathcal{L}(\psi_1, \psi_2) = \sum_{a=1,2} \bar{\psi}_a (\mathrm{i}\gamma^\mu \partial_\mu - m_a)\psi_a - \frac{1}{2} \sum_{ab=1,2} \mu_{ab} \bar{\psi}_a \sigma_{\mu\nu} \psi_b F^{\mu\nu}, \tag{5}$$

where m_a is the mass of the fermion ψ_a and

$$\mu_{ab} = \sum_{\lambda\lambda'=\alpha,\beta} U^{-1}_{a\lambda} M_{\lambda\lambda'} U_{\lambda' b}, \tag{6}$$

is the magnetic moment matrix presented in the mass eigenstates basis. Note that the matrix (μ_{ab}) in Eq. (6) can be non-diagonal, i.e. the transition magnetic moment can have non-zero value, $\mu_{12} = \mu_{21} = \mu \neq 0$.

Let us assume that the electric field vanishes, $\mathbf{E} = 0$. In this case we write down the Dirac-Pauli equation for ψ_a, resulting from Eq. (5), as follows:

$$\mathrm{i}\dot{\psi}_a = \mathcal{H}_a \psi_a + V \psi_b, \quad a, b = 1, 2, \quad a \neq b, \tag{7}$$

where $\mathcal{H}_a = (\boldsymbol{\alpha}\mathbf{p}) + \beta m_a - \mu_a\beta(\boldsymbol{\Sigma}\mathbf{B})$ is the Hamiltonian for the particle ψ_a accounting for the magnetic field, $V = -\mu\beta(\boldsymbol{\Sigma}\mathbf{B})$ describes the interaction of the transition magnetic moment with the external magnetic field and $\mu_a = \mu_{aa}$.

The general solution to Eq. (7) can be presented as follows:

$$\psi_a(\mathbf{r},t) = \int \frac{\mathrm{d}^3\mathbf{p}}{(2\pi)^{3/2}} e^{\mathrm{i}\mathbf{pr}} \sum_{\zeta=\pm 1} \left[a_a^{(\zeta)}(t) u_a^{(\zeta)} \exp\left(-\mathrm{i}E_a^{(\zeta)}t\right) + b_a^{(\zeta)}(t) v_a^{(\zeta)} \exp\left(+\mathrm{i}E_a^{(\zeta)}t\right)\right], \quad (8)$$

The basis spinors $u_a^{(\zeta)}$ and $v_a^{(\zeta)}$, as well as the energy $E_a^{(\zeta)}$, are the exact solutions to the Dirac equations,

$$\mathcal{H}_a u_a^{(\zeta)} = E_a^{(\zeta)} u_a^{(\zeta)}, \quad \mathcal{H}_a v_a^{(\zeta)} = -E_a^{(\zeta)} v_a^{(\zeta)}, \quad (9)$$

accounting for the external magnetic field. The discrete quantum number $\zeta = \pm 1$ describes different polarization states of the fermion ψ_a. The coefficients $a_a^{(\zeta)}$ and $b_a^{(\zeta)}$ in Eq. (8) are in general functions of time.

3 Neutrino spin-flavor oscillations in the transversal magnetic field

In this section we study the evolution of the system in question under the influence of the constant magnetic field directed along the z-axis, $\mathbf{B} = (0,0,B)$.

The basis spinors and energy levels in Eq. (8) can be found in Refs. [4, 10]. The energy as a function of the particle mass and momentum $\mathbf{p} = (p_1, p_2, p_3)$ is

$$E_a^{(\zeta)} = \sqrt{p_3^2 + \mathcal{E}_a^{(\zeta)2}}, \quad \mathcal{E}_a^{(\zeta)} = \mathcal{E}_a - \zeta\mu_a B, \quad \mathcal{E}_a = \sqrt{m_a^2 + p_1^2 + p_2^2}. \quad (10)$$

The basis spinors are expressed in the following form:

$$u_a^{(\zeta)} = \frac{1}{2\sqrt{E_a^{(\zeta)}}} \begin{pmatrix} \phi_a^+ \alpha_a^+ \\ -\zeta\phi_a^- \alpha_a^- e^{\mathrm{i}\varphi} \\ \phi_a^+ \alpha_a^- \\ \zeta\phi_a^- \alpha_a^+ e^{\mathrm{i}\varphi} \end{pmatrix}, \quad v_a^{(\zeta)} = \frac{1}{2\sqrt{E_a^{(\zeta)}}} \begin{pmatrix} \phi_a^+ \alpha_a^- \\ \zeta\phi_a^- \alpha_a^+ e^{\mathrm{i}\varphi} \\ -\phi_a^+ \alpha_a^+ \\ \zeta\phi_a^- \alpha_a^- e^{\mathrm{i}\varphi} \end{pmatrix}, \quad (11)$$

where

$$\phi_a^\pm = \sqrt{1 \pm \zeta m_a/\mathcal{E}_a}, \quad \alpha_a^\pm = \sqrt{E_a^{(\zeta)} \pm \zeta\mathcal{E}_a^{(\zeta)}},$$

and $\tan\varphi = p_2/p_1$.

Using the results of our paper [4] we obtain the right-handed component of the wave function ν_α accounting for the initial condition (2),

$$\begin{aligned} \nu_\alpha^{\mathrm{R}}(x,t) = & \bigg\{ \sin\theta\cos\theta \frac{1}{2\mathrm{i}} \left[\frac{\omega_+}{\Omega_+} \sin(\Omega_+ t) \exp\left(\mathrm{i}\bar{\mu}Bt\right) - \frac{\omega_-}{\Omega_-} \sin(\Omega_- t) \exp\left(-\mathrm{i}\bar{\mu}Bt\right) \right] \\ & + \mathrm{i}\mu B \left[\frac{\sin(\Omega_+ t)}{\Omega_+} \cos^2\theta - \frac{\sin(\Omega_- t)}{\Omega_-} \sin^2\theta \right] \cos(\bar{\mu}Bt) \bigg\} \exp\left(-\mathrm{i}\bar{\mathcal{E}}t + \mathrm{i}kx\right)\kappa_0, \end{aligned} \quad (12)$$

where $\Omega_\pm = \sqrt{(\mu B)^2 + (\omega_\pm/2)^2}$, $\omega_\pm = E_1^\pm - E_2^\pm$, $\bar{\mu} = (\mu_1 + \mu_2)/2$, $\bar{\mathcal{E}} = (\mathcal{E}_1 + \mathcal{E}_2)/2$ and $\kappa_0^{\mathrm{T}} = (1/2)(1,1,1,1)$. Note that to obtain Eq. (12) we approach to the relativistic limit $k \gg m_a$. Eq. (12) is the most general one which accounts for all neutrino magnetic moments. The transition probability for the process $\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}$ can be calculated as $P_{\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}}(t) = |\nu_\alpha^{\mathrm{R}}(x,t)|^2$. It should be also mentioned that Eq. (12) corresponds to neutrino spin-flavor oscillations in the transversal magnetic field since $\mathbf{k} = (k,0,0)$ and $\mathbf{B} = (0,0,B)$.

Let us discuss two applications of Eq. (12). First we consider the case when $\mu_a \gg \mu$, i.e. the magnetic moments matrix in Eq. (6) is close to diagonal. In this situation the transition probability calculated from Eq. (12) is

$$P_{\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}}(t) = \sin^2(2\theta) \left\{ \sin^2(\delta\mu B t) \cos^2(\bar{\mu} B t) + \sin(\mu_1 B t) \sin(\mu_2 B t) \sin^2 [\Phi(k) t] \right\}, \tag{13}$$

where $\delta\mu = (\mu_1 - \mu_2)/2$, $\Phi(k) = \delta m^2/(4k)$ is the phase of vacuum oscillations and $\delta m^2 = m_1^2 - m_2^2$. In Eq. (13) we present the zero order (in μ) contribution to the transition probability. The next order correction can be found in our work [4].

Now we consider the situation when $\mu \gg \mu_a$, i.e. magnetic moments matrix in Eq. (6) with great non-diagonal elements. In this case the transition probability based on Eq. (12) is (see Ref. [4]),

$$P_{\nu_\beta^{\mathrm{L}} \to \nu_\alpha^{\mathrm{R}}}(t) = \cos^2(2\theta) \left(\frac{\mu B}{\Omega} \right)^2 \sin^2(\Omega t), \tag{14}$$

where $\Omega = \sqrt{(\mu B)^2 + \Phi^2(k)}$.

Using Eqs. (13) and (14) one can compute transition probabilities for spin-flavor oscillations of Dirac neutrinos when particles interact with the constant transversal magnetic field. However neutrino oscillations with the most general magnetic moments matrix should be studied on the basis of Eq. (12).

4 Neutrino spin-flavor oscillations in the twisting magnetic field

In this section we examine the evolution of two Dirac neutrinos under the influence of the twisting magnetic field, $\mathbf{B} = B(0, \sin\omega x, \cos\omega x)$, where ω is the frequency of the magnetic field rotation in space. Note that neutrino spin-flavor oscillations in the twisting magnetic field were studied in Refs. [11] in frames of the standard quantum mechanical approach.

For this configuration of the magnetic field we should consider the modified wave functions in Eq. (8), $\psi_a \to \tilde{\psi}_a = \mathcal{U}^\dagger \psi_a$, where $\mathcal{U} = \mathrm{diag}(\mathfrak{U}, \mathfrak{U})$ and $\mathfrak{U} = \cos(\omega x/2) + \mathrm{i}\sigma_1 \sin(\omega x/2)$. The Hamiltonian for the fermions $\tilde{\psi}_a$ appears to be x-coordinate independent (see Ref. [5]). The basis spinors and energy levels in the modified equation (8) were found in Ref. [5]. The energy as a function of the particle mass and momentum, which is directed along the x-axis, $\mathbf{p} = (p, 0, 0)$, is

$$E_a^{(\zeta)} = \sqrt{\mathcal{M}_a^2 + m_a^2 + p^2 - 2\zeta R_a^2}, \tag{15}$$

where $R_a^2 = \sqrt{p^2 \mathcal{M}_a^2 + (\mu_a B)^2 m_a^2}$ and $\mathcal{M}_a = \sqrt{(\mu_a B)^2 + \omega^2/4}$. The basis spinors take the following form in the relativistic limit:

$$u_a^{(\zeta)} = \frac{1}{2\sqrt{2\mathcal{M}_a[\mathcal{M}_a + \zeta\omega/2]}} \begin{pmatrix} \mu_a B + \zeta\mathcal{M}_a + \omega/2 \\ \mu_a B - \zeta\mathcal{M}_a - \omega/2 \\ \mu_a B - \zeta\mathcal{M}_a - \omega/2 \\ \mu_a B + \zeta\mathcal{M}_a + \omega/2 \end{pmatrix},$$

$$v_a^{(\zeta)} = \frac{1}{2\sqrt{2\mathcal{M}_a[\mathcal{M}_a - \zeta\omega/2]}} \begin{pmatrix} \mathcal{M}_a - \zeta\omega/2 - \zeta\mu_a B \\ \mathcal{M}_a - \zeta\omega/2 + \zeta\mu_a B \\ \zeta\omega/2 - \mathcal{M}_a - \zeta\mu_a B \\ \zeta\omega/2 - \mathcal{M}_a + \zeta\mu_a B \end{pmatrix}. \tag{16}$$

Note that the spinors in Eq. (16) satisfy the orthonormality conditions.

First let us study the situation when the magnetic moments matrix in Eq. (6) is close to diagonal, i.e. $\mu_a \gg \mu$. Using the calculations analogous to those in Sec. 3 and the results of our

work [5] we receive the zero order term (in μ) in the expansion of the transition probability as

$$
\begin{aligned}
P_{\nu_\beta^{\mathrm{L}}\to\nu_\alpha^{\mathrm{R}}}(t) = & \frac{\sin^2(2\theta)}{4}\bigg\{\left(\frac{\mu_1 B}{\mathcal{M}_1}\sin\mathcal{M}_1 t - \frac{\mu_2 B}{\mathcal{M}_2}\sin\mathcal{M}_2 t\right)^2 \\
& + 4\frac{\mu_1\mu_2 B^2}{\mathcal{M}_1\mathcal{M}_2}\sin\mathcal{M}_1 t\sin\mathcal{M}_2 t\sin^2[\Phi(k)]\bigg\}.
\end{aligned} \tag{17}
$$

Here $\Phi(k) = \delta m^2/[4(k+\omega/2)]$ is the oscillations phase which now depends on the frequency of the twisting magnetic field. The next order correction in μ can be found in Ref. [5].

Now we study the opposite case – the magnetic moments matrix in Eq. (6) with great non-diagonal elements, $\mu \gg \mu_a$. This situation should be analyzed non-perturbatively. With help of the results of our work [5] we get the following transition probability:

$$
P_{\nu_\beta^{\mathrm{L}}\to\nu_\alpha^{\mathrm{R}}}(t) = (\mu B)^2\left[\cos^2\theta\frac{\sin\Omega_+ t}{\Omega_+} - \sin^2\theta\frac{\sin\Omega_- t}{\Omega_-}\right]^2, \tag{18}
$$

where $\Omega_\pm = \sqrt{(\mu B)^2 + [\Phi(k)\pm\omega/2]^2}$.

Eqs. (17) and (18) allow one to calculate transition probabilities for spin-flavor oscillations of Dirac neutrinos when particles interact with the twisting – or spiral undulator – magnetic field and propagate along the undulator axis since $\mathbf{k} = (k,0,0)$ and $\mathbf{B} = B(0,\sin\omega x,\cos\omega x)$. It should be noted that Eqs. (17) and (18) reproduce the case of the constant transversal magnetic field, i.e. Eqs. (13) and (14), if we set $\omega = 0$ there.

5 Conclusion

We have summarized the results of our studies of the evolution of Dirac neutrinos in external electromagnetic fields. To describe the time evolution of the neutrinos system we have applied the recently developed approach (see Refs. [4, 5]) which is based on the the exact solutions to the Dirac-Pauli equation in an external magnetic field with the given initial condition.

First (Sec. 3) we have studied the dynamics of two mixed Dirac neutrinos with arbitrary magnetic moments matrix in the constant transversal magnetic field. Using earlier obtained solutions to the Dirac-Pauli equation we derive the neutrino wave function exactly accounting for all neutrino magnetic moments [Eq. (12)] and valid for arbitrary magnetic fields. Then we have appled this result for oscillations of neutrinos with various magnetic moments matrices and derived transition probabilities [Eqs. (13) and (14)]. We have discussed another magnetic field configuration in Sec. 4. The evolution of Dirac neutrinos in the twisting magnetic field has been studied there. On the basis of the Dirac-Pauli equation solution in this external magnetic field we have obtained the transition probabilities [Eqs. (17) and (18)] for different magnetic moments matrices.

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References

[1] J. Schechter and J. W. F. Valle, Phys. Rev. D **24**, 1883 (1981); M. B. Voloshin, M. I. Vysotskiĭ, and L. B. Okun', JETP **64**, 446 (1986); Sov. J. Nucl. Phys. **44**, 440 (1986); E. Akhmedov, Phys. Lett. B **213**, 64 (1988); C.-S. Lim and W. J. Marciano, Phys. Rev. D **37**, 1368 (1988).

[2] B. C. Chauhan, J. Pulido and R. S. Raghavan, JHEP **0507**, 051 (2005), hep-ph/0504069; J. Barranco *et al.*, Phys. Rev. D **66**, 093009 (2002), hep-ph/0207326.

[3] E. Kh. Akhmedov and J. Pulido, Phys. Lett. B **553**, 7 (2003), hep-ph/0209192; A. B. Balantekin and C. Volpe, Phys. Rev. D **72**, 033008 (2005), hep-ph/0411148.

[4] M. Dvornikov and J. Maalampi, hep-ph/0701209.

[5] M. Dvornikov, hep-ph/0708.2328.

[6] M. Dvornikov, Phys. Lett. B **610**, 262 (2005), hep-ph/0411101; in *Proceedings of the IPM school and conference on Lepton and Hadron Physics, Tehran, 2006*, ed. by Y. Farzan, eConf C0605151 (2007), hep-ph/0609139; hep-ph/0610047.

[7] M. Dvornikov, Eur. Phys. J. C **47**, 437 (2006), hep-ph/0601156; hep-ph/0708.2975.

[8] M. S. Dvornikov and A. I. Studenikin, Phys. At. Nucl. **64**, 1624 (2001); *ibid.* **67**, 719 (2004); M. Dvornikov, *ibid.* **70**, 342 (2007), hep-ph/0410152; E. Kh. Akhmedov and M. Yu. Khlopov, Sov. J. Nucl. Phys. **47**, 689 (1988); A. Egorov, A. Lobanov, and A. Studenikin, Phys. Lett. B **491**, 137 (2000), hep-ph/9910476; H. Gies and R. Shaisultanov, Phys. Lett. B **480** 129 (2000), hep-ph/0009342.

[9] E. Elizalde, E. J. Ferrer and V. de la Incera, Phys. Rev. D **70**, 043012 (2004), hep-ph/0404234; E. J. Ferrer and V. de la Incera, Int. J. Mod. Phys. A **19**, 5385 (2004), hep-ph/0408108.

[10] I. M. Ternov, V. G. Bagrov, and A. M. Khapaev, JETP **21**, 613 (1965).

[11] A. Yu. Smirnov, Phys. Lett B **260**, 161 (1991); E. Kh. Akhmedov, P. I. Krastev, and A. Yu. Smirnov, Z. Phys. C **48**, 701 (1991); E. Kh. Akhmedov, S. T. Petcov, and A. Yu. Smirnov, Phys. Rev. D **48**, 2167 (1993), hep-ph/9301211.